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Pseudospectral Approximation to Two-Dimensional Turbulence

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Pseudospectral (collocation) approximation is shown to give results virtually indistinguishable from spectral (Galerkin) approximation in the case of accurate simulations of two-dimensional incompressible turbulence, despite the presence of aliasing terms. Our results suggest the wider application of the efficient and versatile pseudospectral method.

It has recently been shown [1] that pseudospectral approximation [2, Sec. 7] and spectral approximation [2, Sec. 2] give similarly accurate results for some simple model problems, despite the inclusion of aliasing terms in pseudospectral approximation. The results of [1] suggest that the common reference to aliasing "error" is a misnomer. In the present paper, we show that aliasing does not lead to gross errors in the case of two-dimensional, homogeneous, incompressible turbulence; an extensive presentation of the results of spectral simulations of two-dimensional turbulence is given elsewhere [3].

Pseudospectral approximation uses truncated spectral series to obtain approximations to derivatives and imposes the differential equation at selected discrete points. On the other hand, spectral approximation attempts to distribute the error more uniformly by making the error in the differential equation orthogonal to the retained spectral functions [2]. The pseudospectral idea is identical to that of collo-

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cation [4], except that the term pseudospectral is applied to emphasize the connection with spectral series; if a problem has an infinitely differentiable solution, then pseudospectral (and spectral) approximation with an N term spectral series converges faster than any power of $1/N$ as $N \to \infty$. The latter property, or the closely related one of absence of phase errors, leads to the substantial accuracy [5] and efficiency gains of spectral approximations over finite-difference approximations. The term pseudospectral also deemphasizes the link of the present collocation method with recently proposed finite-element collocation methods [6].

We illustrate the technique of pseudospectral approximation for the vorticity-streamfunction formulation of two-dimensional incompressible flow:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (\mathbf{u}\zeta) = \nu \nabla^2 \zeta, \tag{1}$$

$$\zeta = -\nabla^2 \psi, \tag{2}$$

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right), \tag{3}$$

where ψ is the streamfunction, ζ is the vorticity, $\mathbf{u}$ is the incompressible velocity field, and ν is the kinematic viscosity.

With periodic boundary conditions of period 2π in both space directions, it is appropriate to expand the dependent variables ψ, ζ, $\mathbf{u}$ in Fourier series. Pseudospectral approximation to (1)–(3) is gotten by using the Fourier series to obtain accurate values for spatial derivatives at the N^2 equally spaced (collocation) points

$$\mathbf{x}_{jk} = \frac{2\pi}{N}(j, k) \qquad (j, k = 0, \ldots, N-1).$$

In other words, if

$$\zeta(\mathbf{x}_{jk}) = \sum_{\|\mathbf{p}\| < \frac{1}{2}N} \hat{\zeta}(\mathbf{p}) \exp(i\mathbf{p} \cdot \mathbf{x}_{jk}), \tag{4}$$

the pseudospectral approximation to $\nabla\zeta$ at $\mathbf{x}_{jk}$ is

$$\nabla\zeta(\mathbf{x}_{jk}) = \sum_{\|\mathbf{p}\| < \frac{1}{2}N} i\mathbf{p}\hat{\zeta}(\mathbf{p}) \exp(i\mathbf{p} \cdot \mathbf{x}_{jk}). \tag{5}$$

For now $\| \mathbf{p} \| < \frac{1}{2}N$ means $-\frac{1}{2}N < p_1, p_2 < \frac{1}{2}N$. The transforms (4), (5) are efficiently implementable using the fast Fourier transform algorithm [7]. Periodicity is not an essential feature of the pseudospectral procedure; other boundary conditions (e.g., no-slip [2]) and geometries (e.g., spheroidal [8]) can be handled efficiently provided care is taken in choosing the appropriate spectral series.

Pseudospectral approximation is completed by specifying that products like $\mathbf{u}\zeta$ are evaluated locally in physical space so that $\mathbf{u}\zeta(\mathbf{x}_{jk}) = \mathbf{u}(\mathbf{x}_{jk})\, \zeta(\mathbf{x}_{jk})$. It is here

that aliasing interactions are introduced [2]; spectral (Galerkin) approximation removes the aliasing interactions in the evaluation of products of grid-point functions. In pseudospectral approximation, spectral series are used for evaluation of derivatives, while local physical space products are computed locally in physical space. The overriding principle of pseudospectral approximation is that nonlocal operations like finite-difference approximations to physical-space derivatives and convolution-sum approximations to the spectral transform of physical-space products are avoided.

The pseudospectral approximation may be susceptible to numerical instability due to the inclusion of aliasing interactions [9]. However, aliasing instability is only symptomatic of breakdown of the corresponding solution to the continuum equations. A variety of analytical examples show that aliasing instability only appears when the solution to the continuum problem develops some structure like a shock or a shear layer that can not be resolved on the grid used for the simulation. *In pseudospectral approximation, insufficient resolution may be evidenced either as aliasing instability or as oscillations emanating from the flow singularity, while in spectral approximation insufficient resolution is always evidenced by oscillations.* Spectral schemes conserve quadratic integrals like kinetic energy in the absence of viscous damping and time-differencing errors, so aliasing instability is impossible. Analogous behavior occurs in dissipation-free shock calculations [10] where oscillations appear when there is insufficient grid resolution, no matter how small the time steps.

One advantage of applying pseudospectral approximation to the vorticity equation written in the conservative form (1) is that aliasing instability is prohibited since the quadratic kinetic energy integral, $\frac{1}{2}\int |\mathbf{u}|^2 \, d\mathbf{x}$, is conserved in the absence of viscous dissipation and time-differencing errors. This follows immediately from the fact that pseudospectral approximation to (1)–(3) is equivalent to pseudospectral approximation to the primitive form of the two-dimensional Navier–Stokes equations written in rotation form [5, Eq. (3.4)]; see also the argument given at the end of [5, Sec. 4]. Notice that, in contrast to spectral approximation, pseudospectral approximation is different if (1) is written in nonconservative form; in the latter case, there are no apparent quadratic integrals so aliasing instability is possible. However, despite this possibility, there do not appear to be differences larger than several percent between various pseudospectral approximations, so long as the approximations are compared when either gives an accurate simulation of a solution of (1)–(3). Nevertheless, we recommend application of pseudospectral approximation in a form that is free of aliasing instability, just because these forms seem to give slightly more accurate results.

A simple test problem for the pseudospectral method is given by the 'color' problem of advection by uniform rotation: the dynamical equation is (1) with $\nu = 0$, $\mathbf{u} = (-\Omega y, \Omega x)$ where Ω is the (constant) rotation rate, and ζ a scalar no

longer related to **u** by (2) and (3). We follow the precise conditions of [5, Sec. 2]; other references are given in [5]. The initial values for ζ are zero except within the (discrete grid) constant-ζ contours shown in Fig. 1(a). A similar calculation was

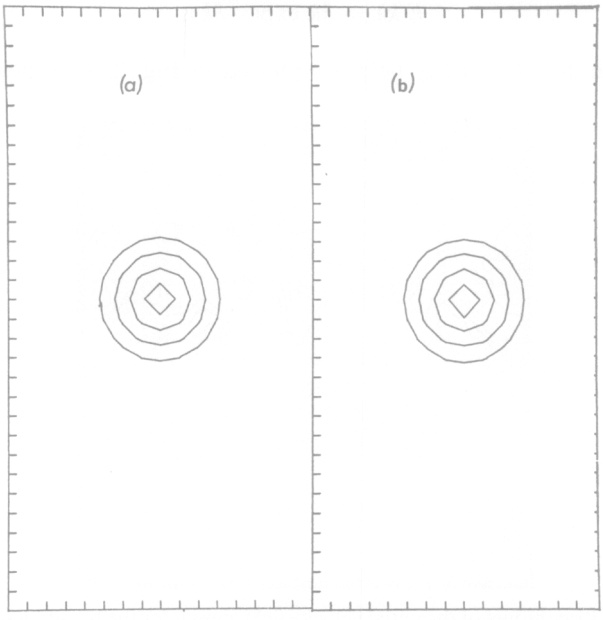

FIG. 1. Pseudospectral approximation on a 32×32 space grid ($N = 32$) for the color problem (advection by uniform rotation). (a) Initial contours of ζ ($\zeta = 0.2, 0.4, 0.6, 0.8$). (b) Contours after one full revolution of underlying rotation field. Time steps were 1/1600 of rotation period to ensure negligible time-differencing errors.

reported previously in [5, Sec. 4(v)] but these results are wrong, because of incorrect inclusion of aliasing effects in an existing unaliased (spectral) code. At that time, we thought we observed aliasing instability. However, it is easy to prove that the eigenvalues of the spatial part of (1) for uniform rotation are pure imaginary (cf. [1]), so that instability can only be due to time differencing. We have repeated the calculations and now find that pseudospectral approximation works very well. Figure 1(b) shows the contours after one full revolution of the underlying rotation field, using a 32×32 ($N = 32$) space grid. After one quarter-revolution, the maximum of ζ is 99 % of its initial maximum, and the negative minimum of ζ is 1 % of its initial maximum; after one full revolution, the corresponding percentages are 98 % and 2 %. Overall, these pseudospectral results are at least as good as any given in [5, Figure 2 and Table 1], including spectral approximation on a 32×32 space grid.

More critical tests are given by applying pseudospectral approximation to two-dimensional turbulence [3]. The initial flow field is chosen to be a realization of a statistically homogeneous, isotropic Gaussian ensemble with (isotropic) energy spectrum

$$E(k) = \tfrac{2}{3}k \exp(-\tfrac{2}{3}k);$$

contours of the initial vorticity field are plotted in Fig. 2. For these tests, (1)–(3)

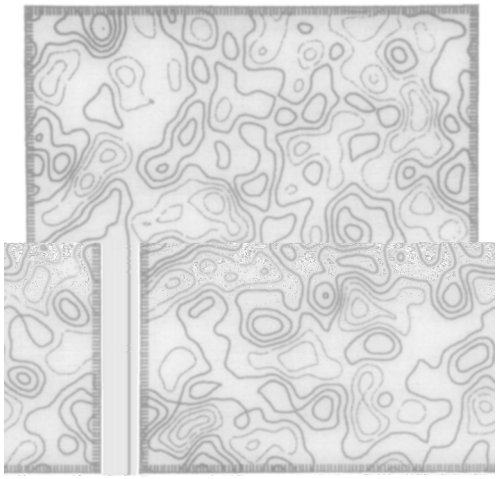

FIG. 2. Vorticity contours of the initial velocity field chosen for the two-dimensional turbulence decay experiments. The velocity field is a realization of a Gaussian, homogeneous, isotropic ensemble with energy spectrum $E(k) = k \exp(-\frac{2}{3}k)$. The field is contoured on the 128×128 space grid used for the numerical experiments.

are solved using both pseudospectral and spectral approximation on a 128×128 ($N = 128$) space grid; the same initial realization is used in all the tests. For these tests, the spectral cutoff $\| \mathbf{p} \| < \frac{1}{2}N$ is now interpreted to mean $| \mathbf{p} |^2 < 3698$, in order to take advantage of the fastest transform method for the spectral calculations [2, Sec. 6]. It turns out that truncation to this circular domain cuts out some aliasing interactions and improves slightly the comparison between spectral and pseudospectral approximations.

Figures 3–5 show the vorticity contours at $t = 1$ for $\nu = 0.0025, 0.0018, 0.001$, respectively, obtained using the pseudospectral method. In the case of Figs. 3, 4, the corresponding figures for the spectral method are *identical* in detail. The spectral results corresponding to Fig. 5 are contoured in Fig. 6; the differences between Figs. 5 and 6 are hardly noticeable. Note that $t = 1$ is well into the evolution of these high Reynolds number flow fields and that the vorticity field is a sensitive measure of small-scale structure. With $\nu = 0.001$, the grid-point

vorticity maximum is 10.03 at $t = 1$ using the spectral method and 10.05 using the pseudospectral method; the corresponding figures for the vorticity minimum are -9.11 and -9.09. With $\nu = 0.0025, 0.0018$, the grid-point vorticity maxima and minima are identical to three decimal places in the spectral and pseudospectral schemes. Further, the two-dimensional skewness factor [3] of the longitudinal vorticity derivative at $t = 1$, $\nu = 0.0018$ is 0.6215 pseudospectrally and 0.6218 spectrally.

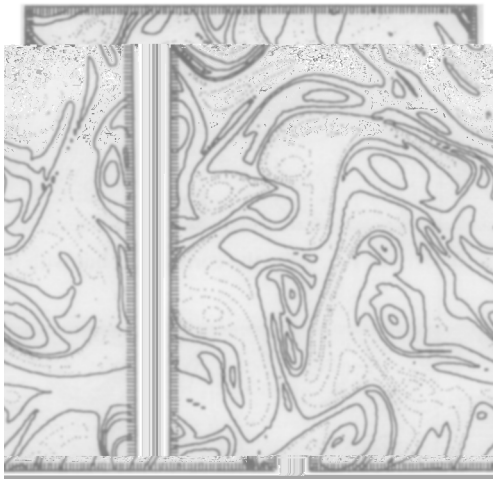

FIG. 3. Vorticity contours at $t = 1$ with $\nu = 0.0025$ using the pseudospectral method with $N = 128$. Spectral calculation with $N = 128$ gives nearly identical contours.

FIG. 4. Same as Fig. 3 but with $\nu = 0.0018$.

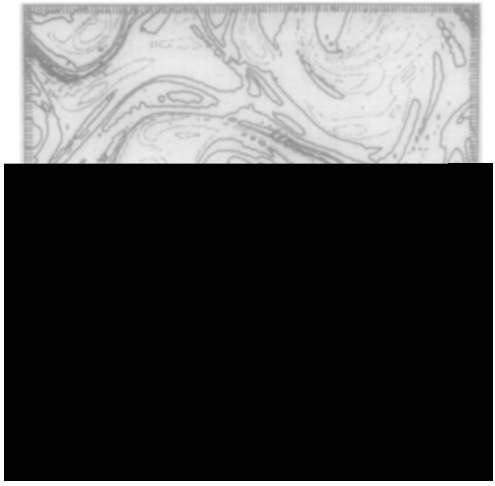

FIG. 5. Same as Fig. 3 but with $\nu = 0.001$. See Fig. 6 for the corresponding spectral results.

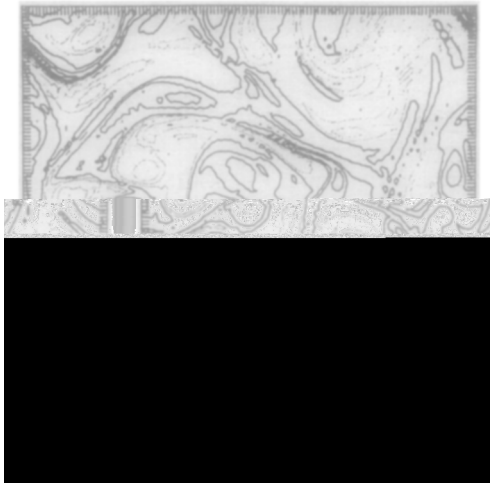

FIG. 6. Vorticity contours at $t = 1$ with $\nu = 0.001$ using the spectral method with $N = 128$.

The results of [3] show that: (i) the spectral simulation with $\nu = 0.0025$ is virtually error-free at $t = 1$, including all the detail scale motions shown in Figure 3; (ii) the spectral results with $\nu = 0.0018$ are in moderate error on scales of less than about 3 grid intervals but are quite accurate on larger scale features *and* phases; and (iii) the spectral results with $\nu = 0.001$ are accurate for scales larger than about 5 grid intervals. For $\nu \lesssim 0.001$, the flow shows signs of incipient

vorticity equipartition [11] and accurate turbulence simulations are not to be expected.

The remarkable resemblance of Figs. 3–6 despite the variation of the Reynolds number by a factor exceeding 2 is discussed in detail in [3].

We conclude that pseudospectral and spectral approximation are equally accurate when faithful simulations of the continuum equations are sought. However, pseudospectral approximation has the advantages that it is (i) *at least* a factor two more efficient than spectral approximation [2], and (ii) applicable to a wide variety of problems with complicated nonlinearities in complicated geometries.

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